ELSEVIER

Discrete Mathematics 131 (1994) 67–79

DISCRETE MATHEMATICS

New classes of Berge perfect graphs

C. De Simone*, A. Galluccio

Istituto di Analisi dei Sistemi ed Informatica del CNR, Viale Manzoni 30, 00185 Roma, Italy

Received 14 November 1990; revised 9 June 1992

Abstract

In this paper we prove the validity of the Strong Perfect Graph Conjecture for some classes of graphs described by forbidden configurations. Three different kinds of techniques are used: the first is the well-known *star-cutset* technique, the second involves a *clique-reduction* operation, and the third is based on a new equivalence of the Strong Perfect Graph Conjecture.

1. Introduction

We assume familiarity with basic notions of graph theory (see, for instance, [1, 3]). A *clique* in a graph is any set of pairwise adjacent nodes; a *stable set* is any set of pairwise nonadjacent nodes. As usual, $\alpha(G)$ denotes the largest size of a stable set in a graph G, $\omega(G)$ denotes the largest size of a clique in G, $\bar{G}$ denotes the complement of G, and $N(u)$ denotes the set of all nodes in G that are adjacent to a node u of G. We denote by $P_k=(v_1, v_2, \dots, v_k)$ the graph with nodes $v_1, \dots, v_k$ and edges $v_i v_{i+1}$ $(i=1, \dots, k-1)$.

Berge proposed to call a graph *perfect* if, for each of its induced subgraph F, the chromatic number of F equals $\omega(F)$. A graph is *minimal imperfect* if it is not perfect but all of its proper induced subgraphs are perfect. Berge conjectured that every minimal imperfect graph is a chordless cycle whose number of nodes is odd and at least five (*odd hole*) or its complement (*odd anti-hole*). This conjecture is known as the *Strong Perfect Graph Conjecture* (see [2] for an extended survey); it remains unsettled. Berge also conjectured that a graph is perfect if and only if its complement is perfect. This was proved by Lovász [6] and it is known as the Perfect Graph Theorem. Lovász [6] also proved that every minimal imperfect graph has $\alpha(G)w(G)+1$ nodes. Chvátal proposed to call a graph *Berge* if none of its induced subgraphs is an odd hole or the

* Corresponding author.

SSDI 0012-365X(92)00531-O

complement of an odd hole. In this terminology, the Strong Perfect Graph Conjecture states that a graph is perfect if and only if it is Berge. Note that a minimal imperfect graph G is Berge if and only if $\alpha(G)\geqslant 3$ and $\omega(G)\geqslant 3$.

A graph G is said to be *partitionable* if there exist integers α and ω such that

— $\alpha\geqslant 2$, $\omega\geqslant 2$,

— for every node v of G there exist partitions of $G-v$ into ω stable sets of size α and α cliques of size ω.

An immediate consequence of Lovász's characterization of perfect graphs is that every minimal imperfect graph G is partitionable with $\alpha=\alpha(G)$ and $\omega=\omega(G)$, and that G is minimal imperfect if and only if its complement is minimal imperfect.

Padberg [10] has shown that every minimal imperfect graph $G=(V,E)$ with $|V|=n$, $\alpha(G)=\alpha$, and $\omega(G)=\omega$ has the following three properties:

— every node is in precisely α stable sets of size α and in precisely ω cliques of size ω;

— G has precisely n stable sets of size α and precisely n cliques of size ω;

— the n stable sets can be enumerated as $S_1, S_2, \ldots, S_n$ and the n cliques can be enumerated as $C_1, C_2, \ldots, C_n$ in such a way that $S_i\cap C_j=\emptyset$ if and only if $i=j$.

A *star-cutset* in a graph G is a nonempty set S of nodes such that the graph $G-S$ is disconnected and such that some node in S is adjacent to all the remaining nodes in S.

Star-Cutset Lemma (Chvátal [4]). *No minimal imperfect graph has a star-cutset.*

Two nodes in a graph are called *twins* if each of the remaining nodes is adjacent to either both of them or none; clearly the Star-Cutset Lemma implies that no minimal imperfect graph has twins. Two nodes in a graph are called *antitwins* if each of the remaining nodes is adjacent to precisely one of these two.

Antitwin Lemma (Olariu [8]). *No minimal imperfect graph contains antitwins.*

Let $G=(V,E)$ be a graph; for every subset K of V, define the graph $G|K=(V|K, E|K)$ by $V|K=V-K$ and

$$E|K=E-\{uv\in E:\ u,v\in K\}\cup\{uv\notin E:\ u,v\notin K,\ N(u)\cup N(v)\supseteq K\}.$$

A *stable crown* of a clique K in a graph $G=(V,E)$ is a subset Z of $V-K$ with the following three properties: (i) Z is a stable set in G of size at least three; (ii) every node in K is adjacent to some node in Z; (iii) for every node u in Z there exists a node v in K for which $N(v)\cap Z=\{u\}$.

Sassano [12] called a clique K in a graph G *reducible* if K is maximal (with respect to set-inclusion) and has no stable crown.

Lemma 1.1 (Sassano [12]). *Let G be a minimal imperfect graph with $\alpha(G)\geqslant 3$. Then, for every reducible clique K of size $\omega(G)$ the graph $G\backslash K$ is imperfect.*

An immediate consequence of Lemma 1.1 is the following.

Corollary 1.2. *Let $\mathscr{G}$ be a hereditary class of Berge graphs. If each minimal imperfect graph G in $\mathscr{G}$ has a reducible clique of size $\omega(G)$ such that $G\backslash K\in\mathscr{G}$, then every graph in $\mathscr{G}$ is perfect.*

Proof. Assume the contrary: there exists a graph G in $\mathscr{G}$ that is not perfect. Without loss of generality, we can assume that G has as few nodes as possible, and so G is minimal imperfect. Since G is Berge, $\alpha(G)\geqslant 3$ [13]. By assumptions, G has a reducible clique K such that $G|K\in\mathscr{G}$. But Lemma 1.1 implies that $G|K$ is imperfect, contradicting the minimality assumption. □

We shall say that an edge uv of a graph G belongs to some triangle if there exists a node of G that is adjacent to both u and v. The following theorem shows that every minimal imperfect Berge graph G contains a spanning subgraph H with the following two properties: (i) H is minimal imperfect and Berge, (ii) each edge of H belongs to some triangle.

Theorem 1.3. *Let e be an edge of a minimal imperfect Berge graph G. If e belongs to no triangle, then $G-e$ is minimal imperfect and Berge.*

Proof. Let u and v be the endpoints of the edge e. Write $\alpha=\alpha(G)$, and $\omega=\omega(G)$. Since G is Berge, $\alpha\geqslant 3$ and $\omega\geqslant 3$. Since e belongs to no triangle, $\omega(G-e)=\omega$. Moreover, $\alpha(G-e)=\alpha$: if $\alpha(G-e)=\alpha+1$ then there exists a stable set in G, say S, of size α containing v, such that u is adjacent to no node in $S-\{v\}$. But then, all cliques of size ω containing u would be disjoint from S, contradicting the assumption that G is a minimal imperfect Berge graph. It follows that $\alpha(G-e)\geqslant 3$ and $\omega(G-e)\geqslant 3$.

Now, we shall prove that $G-e$ is minimal imperfect. We have

$$|G-e|=|G|=\alpha(G-e)\omega(G-e)+1,$$

and so $G-e$ is imperfect. If $G-e$ is not minimal imperfect, then $G-e$ contains a proper induced subgraph F' that is minimal imperfect. Clearly, F' contains both u and v. Let F be the graph obtained from F' by joining u and v. Since G is minimal imperfect and since F is a proper induced subgraph of G, it follows that F is perfect, and so $|F|\leqslant\alpha(F)\omega(F)$. Since e belongs to no triangle, $\omega(F')=\omega(F)$. Hence

$$|F'|=\alpha(F')\omega(F')+1=\alpha(F')\omega(F)+1\geqslant\alpha(F)\omega(F)+1,$$

and so $|F|=|F'|\geqslant\alpha(F)\omega(F)+1$, contradicting the assumption that F is perfect. Hence, $G-e$ is minimal imperfect; moreover, $G-e$ is Berge since $\alpha(G-e)\geqslant 3$ and $\omega(G-e)\geqslant 3$. □

Consider the following conjecture:

$$\begin{array}{l}\textit{G is a minimal imperfect graph if and only if each edge}\\ \textit{of G or each edge of its complement belongs to no triangle.}\end{array} \qquad (*)$$

Clearly, every odd hole and its complement satisfy $(*)$; moreover if $(*)$ holds then every minimal imperfect graph G has $\omega(G)=2$ or $\alpha(G)=2$, and so G or its complement is an odd hole. Hence the Strong Perfect Graph Conjecture is equivalent to conjecture $(*)$. In fact, we can say something more.

Theorem 1.4. *The Strong Perfect Graph Conjecture is true if and only if every minimal imperfect graph or its complement has an edge that belongs to no triangle.*

Proof. (*Only if*) Trivial.

(*If*) If the Strong Perfect Graph Conjecture is false, then there exists a minimal imperfect Berge graph G. By the assumption, G or $\bar{G}$ has some edge that belongs to no triangle. Let F be the graph obtained from G by deleting all edges that belong to no triangle. Theorem 1.3 implies that F is minimal imperfect and Berge. Let H be the graph obtained from F by adding an edge between each pair of nonadjacent nodes u, v of F having the property that every other node of F is adjacent to at least one of them. Again, Theorem 1.3 implies that H is minimal imperfect and Berge (uv is an edge of $\bar{F}$ that belongs to no triangle). We claim that each edge of H belongs to some triangle. To prove the claim, let uv be an edge of H. If uv is an edge of F, by construction it belongs to a triangle; otherwise, the edge uv has been created since every other node of F was adjacent to at least one of u, v. Now, the Antitwin Lemma implies that there exists at least one node of F that is adjacent to both u and v, and so uv belongs to some triangle. But then H is a minimal imperfect Berge graph with the property that every edge of H and every edge of $\bar{H}$ belongs to some triangle, contradicting the assumption that every minimal imperfect graph has an edge that belongs to no triangle. □

Note that Theorem 1.4 allows us to avoid the *each* feature in $(*)$.

When F is a graph, an F-free graph is a graph with no subgraph isomorphic to F. (All our subgraphs are induced.) In the following sections we shall use three different techniques to derive new classes of Berge perfect graphs. The first technique is based on the Star-Cutset Lemma, the second one on Corollary 1.2, and the third one on Theorem 1.4.

2. The star-cutset technique

A popular way of proving that all graphs in a special hereditary class $\mathscr{G}$ of Berge graphs are perfect consists of showing that every minimal imperfect graph in $\mathscr{G}$ has a star-cutset. We shall prove a result of this kind where $\mathscr{G}$ is the class of those Berge graphs defined through two forbidden graphs. For this purpose, we first establish a property of every minimal imperfect Berge graph. To specify this property, we need three more definitions. A *diamond* is the graph with nodes a, b, c, d and edges ab, bc, ac, cd, bd; an *odd apple* is the union of an odd hole and a node that is adjacent to precisely one node in the hole (Fig. 1); *substitution* of a graph G_1 for a node v of a graph G_2 consists of taking a disjoint union of G_1 and $G_2 - v$, and adding an edge between every node of G_1 and every node of $G_2 - v$ that was a neighbour of v in G_2.

In [5] it was shown that every graph G has precisely one of the following two properties:

(a) G can be obtained from bipartite graphs, graphs with no complement of a diamond, and odd holes by repeated substitutions,

(b) G contains an odd apple or one of the graphs F_i ($i = 1, \ldots, 6$) in Fig. 2.

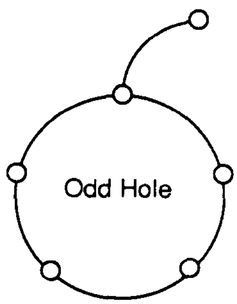


Fig. 1.

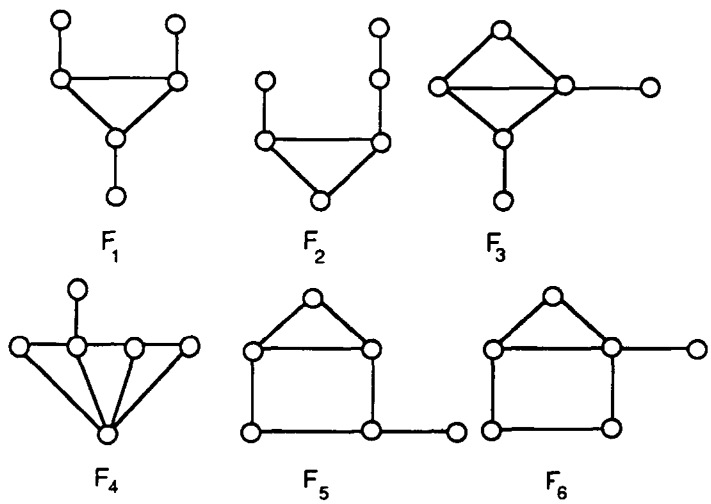


Fig. 2.

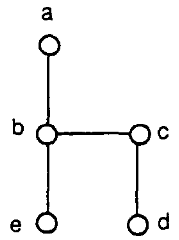


Fig. 3.

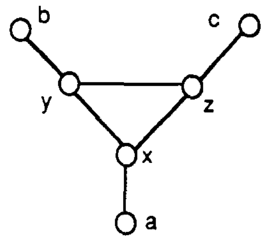


Fig. 4.

Since every minimal imperfect Berge graph does not satisfy property (a) and it contains no odd apple, it follows that

every minimal imperfect Berge graph contains one of $F_1, \ldots, F_6$. (1)

A *chair* is the graph with nodes a, b, c, d, e and edges ab, bc, cd, be (Fig. 3).

Theorem 2.1. *Every chair-free and* F_2*-free Berge graph is perfect.*

Proof. Suppose the contrary: there exists a minimal imperfect chair-free and F_2-free Berge graph G. Since F_3, F_4, F_5, and F_6 contain a chair, (1) implies that G contains an F_1 as induced subgraph. We shall prove that G has a star-cutset. In fact, we shall prove that every chair-free and F_2-free graph that contains an F_1 has a star-cutset.

For this purpose, let $G=(V,E)$ be a chair-free and F_2-free graph that contains the graph in Fig. 4. If the set $\{x\} \cup N(x) - \{a\}$ is a star-cutset, we are done. Otherwise, there is a chordless path P joining a and b whose inner nodes are all nonadjacent to x. Let u be the node of P that is adjacent to a.

If $uy \notin E$ then $ub \notin E$, else u, a, x, y, b would induce a C_5 in G; but then $uz \in E$ (otherwise x, y, z, a, u, b would induce an F_2) and $uc \in E$ (otherwise u, z, y, b, c would induce a chair), and so u, z, c, y, b, a induce an F_2 in G, a contradiction. Hence, $uy \in E$ and, similarly, $uz \in E$. If $ub \in E$ then $uc \in E$, else a, u, z, c, b induce a chair in G; but then c, u, a, x, b induces a chair in G, a contradiction. It follows that $ub \notin E$ and, similarly, $uc \notin E$.

Now, let w be the node of P adjacent to u. If w is adjacent to z, then it is also adjacent to both c and y: w is adjacent to c, otherwise c, z, x, a, w would induce a chair; and w is adjacent to y, otherwise a, u, w, c, y would induce a chair. But then, $wb \in E$, else b, y, w, c, x would induce a chair, and so z, w, c, x, a, b induce an F_2, a contradiction. It follows that w is not adjacent to z and, similarly, w is not adjacent to y. But then, it is easy to verify that the graph induced by w, u, z, c, a, y contains a chair, again a contradiction. □

Observe that the class of chair-free and F_2-free Berge graphs contains the class of chair-free and P_5-free Berge graphs which were proved to be perfect by Olariu [9].

Theorem 2.2. *Every P_5-free and $K_{2,3}$-free Berge graph is perfect.*

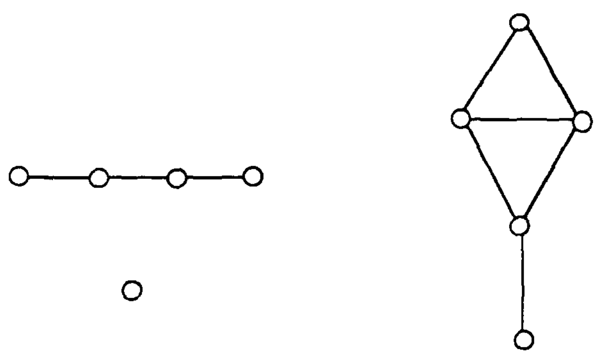
Proof. Suppose the contrary: there exists a minimal imperfect Berge graph G which is P_5-free and $K_{2,3}$-free. We shall show that G has a star-cutset. Since claw-free Berge graphs are perfect [11], it follows that G contains a claw. Let u, x, y, z induce a claw in G with edges ux, uy, uz. Since the set $\{u\} \cup N(u) - \{x, y\}$ is not a star-cutset, it follows that there is a chordless path P joining x and y whose inner nodes are all nonadjacent to u. Since G is Berge, P has an even number of edges; since G is P_5-free, P has exactly two edges.

Let v be the node of P joining x and y. Since G is $K_{2,3}$-free, $vz \notin E$. A similar argument shows that there is a path $P' = (y, w, z)$ such that w is adjacent to neither u nor x. But then the graph induced by v, x, u, z, w is either a P_5 (if $vw \notin E$) or a C_5 (otherwise). $\square$

3. The clique-reduction technique

By Corollary 1.2, to prove that all graphs in a special class $\mathscr{G}$ of Berge graphs are perfect, it suffices to show that for every minimal imperfect graph G in $\mathscr{G}$ there exists a reducible clique of size $\omega(G)$ such that $G|K \in \mathscr{G}$. We shall show that this technique is successful when $\mathscr{G}$ is defined through two forbidden graphs with five nodes: the graphs G_0 and G_1 in Fig. 5.

For this purpose, we need one more definition. Let $G = (V, E)$ be a graph and let K be a clique of size $\omega(G)$ of G. We shall say that an edge uv of the graph $G|K = (V|K, E|K)$ is a *false* edge if $uv \notin E$. The set of all false edges of $G|K$ will be denoted by F.



G_0 G_1

Fig. 5.

Lemma 3.1. *Let G be a G_0-free and G_1-free graph with no twins, let K be a clique of size $\omega(G)$ in G, and let uv be a false edge of $G|K$. Then the following three properties are satisfied*:

$$|N(u)\cap K|=|N(v)\cap K|=|K|-1; \tag{2}$$

no node w of $G|K$ is adjacent to neither u nor v in $G|K$; (3)

for every node w of $G|K$ if $vw\in F$ then $uw\in F$. (4)

Proof. Note that, since K is a maximum clique, for every node in $G|K$ there exists a node in K which is nonadjacent to it; moreover, for every pair of nodes in $V-K$ which are nonadjacent in $G|K$, there exists a node in K which is adjacent to neither of them in G.

To show that (2) is correct, assume the contrary: there exist two nodes in K, say x and y, such that $ux\notin E$ and $uy\notin E$, and so $vx\in E$ and $vy\in E$ (since $uv\in F$). But then, since there exists a node in K, say z, such that $vz\notin E$ (and so $uz\in E$), it follows that u,v,x,y,z induce a G_1 in G, a contradiction. Hence, $|N(u)\cap K|=|K|-1$; similarly, $|N(v)\cap K|=|K|-1$, and so (2) follows.

If (3) is not true then there exists a node w of $G|K$ such that $uw\notin E|K$ and $vw\notin E|K$. Now, $uw\notin E|K$ implies that there exists a node in K, say x, such that $ux\notin E$ and $wx\notin E$, and so $vx\in E$; moreover, $vw\notin E|K$ implies that there exists a node in K, say y, such that $vy\notin E$ and $wy\notin E$, and so $uy\in E$. But then u,y,x,v,w induce a G_0 in G, a contradiction.

Finally, to show that (4) is correct, assume the contrary: there exists a node w in $V-K$ such that $vw\in F$ and $uw\notin F$. Note that (2) implies that there exists a unique node in K, say x, which is nonadjacent to u (and so $vx\in E$) and that there exists a unique node in K, say y, which is nonadjacent to v (and so $uy\in E$ and $wy\in E$). Since $uw\notin F$, it follows that either $uw\in E$ or $uw\notin E|K$.

If $uw\in E$ then $wx\notin E$ (otherwise u,v,w,x,y induce a G_1 in G). But then, (2) implies that $N(u)\cap K=N(w)\cap K$, and so $\{u,w\}\cup(K-\{x\})$ is a clique in G of size $\omega(G)+1$, a contradiction. Hence every triple i,j,k of nodes of $G|K$ satisfies the following property:

$$\textit{if } ij\in F \quad \textit{and} \quad jk\in F \textit{ then } ik\notin E. \tag{5}$$

Assume that now $uw\notin E|K$. Since both u and w are not adjacent to some node in K, property (2) implies that $wx\notin E$ and that $N(u)\cap K=N(w)\cap K$. Since by assumption, G has no twins, there exists in G a node, say z such that $zw\in E$ and $uz\notin E$. Note that property (5) (with i,j,k replaced by z,v,w) implies that $vz\notin F$, and so either $vz\in E$ or $vz\notin E|K$.

If $vz\notin E|K$ then (3) implies that $uz\in F$ (since, by assumption, $uz\notin E$); but then $zx\in E$ (since $ux\notin E$), and so u,v,x,z,w induce a G_0 in G, a contradiction. It follows that $vz\in E$, and so (5) (with i,j,k replaced by z,u,v) implies that $uz\notin F$, and so $uz\notin E|K$ (since, by assumption, $uz\notin E$). But then $zx\notin E$ (since $ux\notin E$), and so u,x,v,z,w induce a G_0 in G, again a contradiction. □

Theorem 3.2. *Let $\mathscr{G}$ be the class of Berge graphs that are G_0-free and G_1-free. Then for every minimal imperfect graph G in $\mathscr{G}$ and for every clique K of size $\omega(G)$ of G, $G|K \in \mathscr{G}$.*

Proof. Let $G=(V, E)$ be a minimal imperfect graph in $\mathscr{G}$, and let K be any clique of size $\omega(G)$ in G. We claim that K is reducible. Indeed, if K is not reducible it has a stable crown Z, and so there exist nodes u_1, u_2, u_3 in K and nodes z_1, z_2, z_3 in Z such that $u_i z_j \in E$ if and only if $i=j$. But then z_1, u_1, u_2, z_2, z_3 induce a G_0 in G, a contradiction. Hence, by Corollary 1.2, we only need verify that $G|K \in \mathscr{G}$.

Now, if the set F of all false edges of $G|K$ is empty, we are done. Hence assume that F is not empty.

Since G is minimal imperfect, it has no twins, and so property (3) in Lemma 3.1 implies that $G|K$ is G_0-free and it does not contain odd holes. Hence, we only need show that $G|K$ is G_1-free and that it does not contain odd anti-holes. Assume that contrary: $G|K$ contains as induced subgraph the graph H_1 in Fig. 6 or an odd anti-hole H_2.

Case A: *$G|K$ contains H_1*. Since G is G_1-free, some edge of H_1 is false. Lemma 3.1 assures that edges ab, ac, bc, de are not false (by property (3)), and so not both edges bd and cd are false (by property (4)). Without loss of generality, we may assume that bd is false and so cd is not false. Let y be the node of K which is not adjacent to d and let x be the node of K which is not adjacent to b (and so $xd \in E$).

It is easy to see that $ey \notin E$: if this is not the case, then either b, c, d, e, y induce a C_5 in G (if $cy \notin E$) or a, b, c, y, e induce a G_1 in G (if $cy \in E$), a contradiction.

Since $be \notin E|K$, it follows that there exists a node in K which is nonadjacent to both b and e, and so property (2) in Lemma 3.1 assures that $ex \notin E$; similarly, since $ad \notin E|K$,

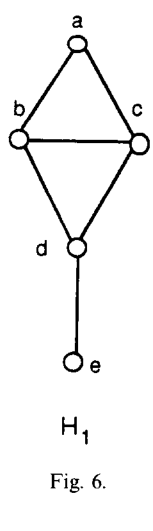


Fig. 6.

it follows that $ay \notin E$. Hence, $ax \in E$ (otherwise a, b, y, x, e induce a G_0 in G) and that $cy \in E$ (otherwise a, c, d, e, y induce a G_0, in G), and so $cx \notin E$ (otherwise y, c, x, d, e induce a G_1 in G).

Now, we claim that $\{a, b, c\} \cup (K - \{x, y\})$ is a clique of size $\omega(G)+1$, which is impossible. To see this, let z be any node in $K - \{x, y\}$ (such a node exists since $\omega(G) \geqslant 3$). Now, property (2) in Lemma 3.1 assures that $bz \in E$ and $dz \in E$, and so $ze \in E$ (for otherwise y, x, z, d, e would induce a G_1 in G). Moreover, $az \in E$ (for otherwise a, x, z, d, e would induce a G_1 in G), and so $cz \in E$ (for otherwise c, a, b, z, e would induce a G_1 in G). Hence the claim is proved.

Case B: *$G|K$ contains H_2*. Write $H_2 = (V', E')$ with

$$V' = \{v_1, v_2, \ldots, v_n\}, \quad n = 2\omega' + 1,$$

$$E' = \{v_i v_j \colon |i - j| \leqslant \omega' - 1 \text{ with subscript arithmetic modulo } n\}.$$

Since G is Berge, some edge of H_2 is false. Let $v_i v_j$ be a false edge of H_2; without loss of generality, we can assume that $i < j < \omega'$. Since $v_{i+\omega'+1} v_j \notin E'$ and $v_{i+\omega'+1} v_{i+1} \notin E'$, property (3) in Lemma 3.1 assures that $j > i+1$. Moreover, since $v_i v_{i+\omega'} \notin E'$ and $v_i v_{i+\omega'+1} \in E'$, property (3) in Lemma 3.1 guarantees that $v_j v_{i+\omega'} \in E$ and $v_j v_{i+\omega'+1} \in E$; similarly, since $v_j v_{j+\omega'} \notin E'$ and $v_j v_{j+\omega'+1} \notin E'$, it follows that $v_i v_{j+\omega'} \in E$ and $v_i v_{j+\omega'+1} \in E$. Again, property (3) in Lemma 3.1 assures that $v_{i+\omega'} v_{i+\omega'+1} \in E$ and $v_{j+\omega'} v_{j+\omega'+1} \in E$.

Note that $v_{j+\omega'} v_{i+\omega'+1} \in E'$, $v_{j+\omega'} v_{i+\omega'} \in E'$, and $v_{j+\omega'+1} v_{i+\omega'+1} \in E'$. Moreover, exactly one of $v_{j+\omega'} v_{i+\omega'+1}$ and $v_{j+\omega'+1} v_{i+\omega'+1}$ is a false edge: both edges cannot be false by property (4) in Lemma 3.1, and if none is false then $v_j, v_{i+\omega'+1}, v_{j+\omega'+1}, v_{j+\omega'}, v_i$ induce a G_1 in G, a contradiction. Similarly, exactly one of $v_{j+\omega'} v_{i+\omega'+1}$ and $v_{j+\omega'} v_{i+\omega'}$ is a false edge.

First, assume that $v_{j+\omega'+1} v_{i+\omega'+1} \in F$, and so $v_{j+\omega'} v_{i+\omega'+1} \in E$ and $v_{j+\omega'} v_{i+\omega'} \in F$. Let x be the node in K which is adjacent to neither v_j, nor $v_{j+\omega'}$, nor $v_{j+\omega'+1}$ in G (such a node exists by property (2) in Lemma 3.1). Since $xv_i \in E$, $xv_{i+\omega'+1} \in E$, and $xv_{i+\omega'} \in E$, it follows that $v_j, v_{i+\omega'}, v_{i+\omega'+1}, x, v_i$ induce a G_1 in G, a contradiction.

Secondly, assume that $v_{j+\omega'+1} v_{i+\omega'+1} \in E$, and so $v_{j+\omega'} v_{i+\omega'+1} \in F$ and $v_{j+\omega'} v_{i+\omega'} \in E$. Let x be the node in K which is adjacent to neither v_j, nor $v_{j+\omega'+1}$, nor $v_{j+\omega'}$. Clearly, $xv_{i+\omega'+1} \in E$ and $xv_i \in E$. If $xv_{i+\omega'} \in E$ then v_j, $v_{i+\omega'}$,'$v_{i+\omega'+1}, x, v_i$ induce a G_1 in G, a contradiction. Hence both $v_{j+\omega'+1}$ and $v_{i+\omega'}$ are not adjacent to x, and so property (3) in Lemma 3.1 assures that $v_{i+\omega'} v_{j+\omega'+1} \notin F$. But then $v_{i+\omega'} v_{j+\omega'+1} \notin E'$ (otherwise $v_j, v_{i+\omega'}, v_{j+\omega'}, v_{j+\omega'+1}, v_i$ induce a G_1 in G), and so $v_j, v_{i+\omega'}, v_{j+\omega'+1}, v_{j+\omega'}, x$ induce a G_0 in G, a contradiction.

It follows that $G|K$ does not contain odd anti-holes, and so the theorem is proved. □

Corollary 3.3. *Every G_0-free and G_1-free Berge graph is perfect.*

4. The triangle technique

We shall say that two nonadjacent nodes in a graph are *witnesses* if every other node is adjacent to at least one of them. Theorem 1.4 provides an alternative way of proving the validity of the Strong Perfect Graph Conjecture for special classes of graphs. It consists of choosing a graph F and then showing that every minimal imperfect Berge graph G that contains no F has one of the following properties:

(i) G has an edge e that belongs to no triangle such that $G-e$ is F-free

(ii) $\bar{G}$ has an edge e that belongs to no triangle such that $\bar{G}-e$ is $\bar{F}$-free.

For instance, consider the class of $\bar{C}_4$-free Berge graphs, which is still unknown to be perfect. Let G be a graph in this class. It is easy to see that the graph G' obtained from G by adding an edge between two witnesses is still a $\bar{C}_4$-free graph. Hence, to show that $\bar{C}_4$-free Berge graphs are perfect, we only need show that every such graph has two witnesses.

Theorem 4.1. *Let G be a $\bar{C}_4$-free minimal imperfect Berge graph. If G does not contain the graph H in Fig. 7, then it has two witnesses.*

Proof. Let $G=(V,E)$ be a $\bar{C}_4$-free minimal imperfect Berge graph that contains no H, and let x and y be two nonadjacent nodes of G with the property that $|N(x)\cup N(y)|$ is as large as possible. Set $A=N(x)-N(y)$, $B=N(y)-N(x)$, and $C=V-N(x)-N(y)$. Clearly, both A and B are not empty: if both A and B are empty then x and y are twins, a contradiction; if only A is empty then the set $y\cup N(y)-\{v\}$ with $v\in B$ is a star-cutset, again a contradiction. Moreover, every node in A is adjacent to every node in B, else G would contain a $\bar{C}_4$. We shall show that the set C is empty, that is the nodes x and y are witnesses.

Assume that C is not empty. Note that

any two nonadjacent nodes in A
have no common neighbour in C. (6)

To see this, assume the contrary: there exist u and v in A such that $uv\notin E$, $uz\in E$ and $vz\in E$ for some z in C. But then u,x,v,z,y would induce an H in G, a contradiction. Let z be a node in C such that $N(z)\cap A\neq\emptyset$; such a node exists since $y\cup N(y)$ is not a star-cutset.

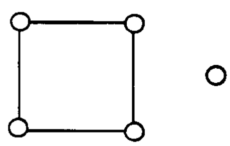


Fig. 7.

Let u be a node in $N(z)\cap A$. Let U be the set of nodes in A that are nonadjacent to u. Note that U is not empty, else u and y would be two nonadjacent nodes with the property that $|N(u)\cup N(y)|>|N(x)\cup N(y)|$, contradicting the maximality assumption. Let v be a node in U. Clearly, (6) implies that $vz\notin E$; moreover, v is adjacent to no node in C: if some node w in C is adjacent to v then (6) implies that z,u,x,v,w induce a P_5 (if $zw\notin E$) or a C_5 (otherwise). Since v was an arbitrary node in U, it follows that every node in U is adjacent to no node in C.

Similarly, there exists a node u' in B such that u' is adjacent to some node z' in C, with possibly $z'=z$, and such that every node in U' (set of nodes in B that are nonadjacent to u') is adjacent to no node in C.

Now, it is easy to see that U induces a clique in G and that $|U|=1$: if two nodes in U, say a,b are nonadjacent then a,b,x,z along with any node in U' induce an H; if U contains at least two nodes then u,z along with any two nodes in U induce a $\bar{C}_4$. Hence, $U=\{v\}$. Moreover, the maximality assumption implies that $N(u)\cap C=\{z\}$. Similarly, U' contains precisely one node, say v', and $N(u')\cap C=\{z'\}$. If C contains more than one node then u,v,x,v' along with any node in $C-\{z\}$ induce an H, a contradiction. Hence, $C=\{z\}$. Clearly,

z is adjacent to precisely one of any two nonadjacent nodes in A. (7)

To see this, let a,b be two nonadjacent nodes in A: (6) implies that z is adjacent to at most one of them; if z is adjacent to neither of them, then a,x,b,v',z induce an H, a contradiction.

Let S be any stable set in G of size $\alpha(G)$ that contains y. Now (7) implies that every stable set in A has size at most two, and so $|S\cap A|\leqslant 2$. But then it is easy to see that $\alpha(G)\leqslant 3$, contradicting the assumption that G is Berge [13]. □

Corollary 4.2. *Every $\bar{C}_4$-free and H-free Berge graph is perfect.*

Proof. Let $\mathcal{G}$ denote the class of $\bar{C}_4$-free and H-free minimal imperfect Berge graphs. Let G be a graph in $\mathcal{G}$ with the property that among all graphs in $\mathcal{G}$ with the same number of nodes, G has as many edges as possible. Since G contains no H, Theorem 4.1 implies that G has two witnesses, say x and y. But the graph G' obtained from G by joining x and y is still $\bar{C}_4$-free and H-free; moreover, Theorem 1.3 and the Perfect Graph Theorem imply that G' is minimal imperfect and Berge, and so $G'\in\mathcal{G}$, contradicting the maximality assumption. □

5. Conclusions

All classes, but one, of Berge perfect graphs that were found in this paper have the property that every graph in this class contains no induced $\bar{C}_4$. Hence, every $\bar{C}_4$-free

minimal imperfect Berge graph has the following properties:
— it contains a chair (by Theorem 2.1);
— it contains a $K_{2,3}$ (by Theorem 2.2);
— it contains the graph H in Fig. 7 (by Corollary 4.2);
— it has no witnesses (by Theorem 4.1).

Acknowledgment

We thank an anonymous referee for helpful suggestions.

References

[1] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).
[2] C. Berge and V. Chvátal, Topics on perfect graphs, Ann. Discrete Math. 21 (1984).
[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillian, London, 1976).
[4] V. Chvátal, Star-cutsets and perfect graphs, J. Combin. Theory B 39 (1985) 189–299.
[5] C. De Simone, On the vertex packing problem, Graphs Combin. 9 (1993) 19–30.
[6] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. 2 (1972) 253–267.
[7] L. Lovász, A characterization of perfect graphs, J. Combin. Theory Ser. B 13 (1972) 95–98.
[8] S. Olariu, No Antitwins in minimal imperfect graphs, J. Combin. Theory Ser. B 45 (1988) 255–257.
[9] S. Olariu, On the strong perfect graph conjecture, J. Graph Theory 12 (1988) 169–176.
[10] M. Padberg, Perfect zero-one matrices, Math. Programming 6 (1974) 180–196.
[11] K.R. Parthasarathy and G. Ravindra, The strong perfect graph conjecture is true for $K_{1,3}$-free graphs, J. Combin. Theory Ser. B 28 (1976) 212–223.
[12] A. Sassano, Reducible cliques and the strong perfect graph conjecture, IASI Tech. Report R.257, March 1989.
[13] A. Tucker, Critical perfect graphs and perfect 3-chromatic graphs, J. Combin. Theory Ser. B 23 (1977) 143–149.